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A biparametric family of four-step sixteenth-order root-finding methods with the optimal efficiency index

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ABSTRACT

A biparametric family of four-step multipoint iterative methods of order sixteen to solve nonlinear equations are developed and their convergence properties are established. The optimal efficiency indices are all found to be $16^{1/5} \approx 1.741101$. Numerical examples as well as comparison with existing methods are demonstrated to verify the developed theory.

1. Introduction

To find a numerical solution of a nonlinear equation $f(x) = 0$, a variety of eighth-order 3-step multipoint iterative methods free from second derivatives have been developed by Bi–Ren–Wu [2], Bi–Wu–Ren [3], Geum–Kim [5], Liu–Wang [7] and Wang–Liu [10]. In 1981, Neta [8] suggested a family of sixteenth-order multipoint iterative methods with the optimal efficiency index [9] of $16^{1/5}$ which are introduced here:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n) + Af(y_n)}{f(x_n) + (A-2)f(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \quad A \in \mathbb{R}, \\ s_n = y_n + \delta_1 f^2(x_n) + \delta_2 f^3(x_n), \\ x_{n+1} = y_n + \theta_1 f^2(x_n) + \theta_2 f^3(x_n) + \theta_3 f^4(x_n), \end{cases} \tag{1.1}$$

where $\delta_2 = -\frac{\phi_y - \phi_z}{F_y - F_z}$, $\delta_1 = \phi_y + \delta_2 F_y$, $\theta_3 = \frac{\Delta_1 - \Delta_2}{F_s - F_y}$, $\theta_2 = -\Delta_1 + \theta_3(F_s + F_z)$, $\theta_1 = \phi_s + \theta_2 F_s - \theta_3 F_s^2$ with $\Delta_1 = \frac{\phi_s - \phi_z}{F_s - F_z}$, $\Delta_2 = \frac{\phi_y - \phi_z}{F_y - F_z}$, $\phi_s = \frac{1}{F_s}\left(\frac{s_n - x_n}{F_s} - \frac{1}{f'(x_n)}\right)$, $\phi_y = \frac{1}{F_y}\left(\frac{y_n - x_n}{F_y} - \frac{1}{f'(x_n)}\right)$, $\phi_z = \frac{1}{F_z}\left(\frac{z_n - x_n}{F_z} - \frac{1}{f'(x_n)}\right)$, $F_s = f(s_n) - f(x_n)$, $F_y = f(y_n) - f(x_n)$ and $F_z = f(z_n) - f(x_n)$.

Notice that the coefficients $\delta_i (i = 1, 2)$ as well as $\theta_i (i = 1, 2, 3)$ are dependent upon the values of $x_n, y_n, z_n, s_n, f(x_n), f(y_n), f(z_n), f(s_n), f'(x_n)$. Such coefficients unfavorably require much computational time. We find below the

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corresponding error equation of (1.1) that Neta [8] did not provide:

$$e_{n+1} = c_2^4[(1+2A)c_2^2 - c_3]^2(5c_2^3 - 5c_2c_3 + c_4)(14c_2^4 - 21c_2^2c_3 + 3c_3^2 + 6c_2c_4 - c_5)e_n^{16} + O(e_n^{17}). \quad (1.2)$$

The main aim is to develop a higher-order method of optimal order consistent with the conjecture of Kung–Traub [6] for complex-valued as well as real-valued nonlinear equations. We assume that $f : \mathbb{C} \to \mathbb{C}$ has a simple root α and is analytic [1] in a region containing α. Introducing constant parameters, we propose a new family of four-step multipoint methods of order sixteen described as follows: for $n = 0, 1, \ldots,$

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - K_f(u_n)\dfrac{f(y_n)}{f'(x_n)}, \\ s_n = z_n - H_f(u_n, v_n, w_n)\dfrac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = s_n - W_f(u_n, v_n, w_n, t_n)\dfrac{f(s_n)}{f'(x_n)}, \end{cases} \quad (1.3)$$

where

$$\begin{aligned} &K_f(u_n) = \frac{1 + \beta u_n - \frac{\beta}{2}u_n^2}{1 + (\beta - 2)u_n - (1 + \frac{5}{2}\beta)u_n^2}, \qquad H_f(u_n, v_n, w_n) = \frac{1 - u_n - \frac{3}{2}v_n - \frac{5}{2}w_n}{1 - 3u_n - \frac{5}{2}v_n + \frac{3}{2}w_n}, \\ &W_f(u_n, v_n, w_n, t_n) \\ &\quad = \frac{1 - u_n - \frac{3}{2}v_n - 3w_n + \frac{3}{2}t_n - \frac{13}{4}v_nw_n + \frac{3}{4}v_n^3 - \frac{1}{4}(\beta^2 - \beta + 8)v_nu_n^4 - \frac{3}{2}t_nu_n^2 + \Omega u_nw_n^2}{1 - 3u_n - \frac{5}{2}v_n + w_n + \frac{1}{2}t_n - \frac{19}{4}v_nw_n - \frac{3}{4}v_n^3 - \frac{1}{4}(\beta^2 - 3\beta + 8)v_nu_n^4 - \frac{9}{2}t_nu_n^2 + (\Omega - \frac{27}{2})u_nw_n^2}, \end{aligned} \quad (1.4)$$

with two constant real parameters β, Ω to be chosen freely, and

$$u_n = f(y_n)/f(x_n), \qquad v_n = f(z_n)/f(y_n), \qquad w_n = f(z_n)/f(x_n), \qquad t_n = f(s_n)/f(z_n). \quad (1.5)$$

Observe that (1.3) requires five new function evaluations for $f(x_n), f(y_n), f(z_n), f(s_n)$ and $f'(x_n)$ per iteration. Hence the optimal efficiency index of $16^{1/5} \approx 1.741101$ for (1.3) will be obtained along with the derivation of the corresponding error equation stating convergence order of sixteen. As a measure of convergence behavior, the values of $|x_n - \alpha|$ as well as CPU times of the proposed method (1.3) will be compared with those of the iterative method (1.1). For typical forms of $W_f(u_n, v_n, w_n, t_n)$, numerical examples will be presented to verify the underlying theory developed in this paper.

2. Convergence analysis

Introducing general parameters $\lambda, \mu, \gamma, a, b, c, d, k_1, k_2, \ldots, k_8, \rho, \Gamma, \Psi, \Delta, \Omega, \hat{\rho}, \hat{\Gamma}, \hat{\Psi}, \hat{\Delta}, \hat{\Omega}$ in (1.4) yields:

$$\begin{aligned} &K_f(u) = \frac{1 + \beta u + \lambda u^2}{1 + (\beta - 2)u + \mu u^2}, \qquad H_f(u, v, w) = \frac{1 + au + bv + \gamma w}{1 + cu + dv + \sigma w}, \\ &W_f(u, v, w, t) = \frac{1 + k_1u + k_2v + k_3w + k_4t + \rho vw + \Gamma v^3 + \Psi vu^4 + \Delta tu^2 + \Omega uw^2}{1 + k_5u + k_6v + k_7w + k_8t + \hat{\rho}vw + \hat{\Gamma}v^3 + \hat{\Psi}vu^4 + \hat{\Delta}tu^2 + \hat{\Omega}uw^2}. \end{aligned} \quad (2.1)$$

The choice of parameters in (2.1) will be made based on the method of undetermined coefficients which can be found in [4,11]. In what follows, Theorem 2.1 describes the convergence analysis on iterative scheme (1.3) with (1.4).

Theorem 2.1. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ has a simple root α and is analytic in a region containing α. Let $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \ldots$. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. If*

$$\begin{aligned} &a = -1, \quad b = -\frac{3}{2}, \quad c = -3, \quad d = -\frac{5}{2}, \quad \gamma = -\frac{5}{2}, \quad \sigma = \frac{3}{2}, \quad \rho = -\frac{13}{4}, \quad \Gamma = \frac{3}{4}, \\ &\Delta = -\frac{3}{2}, \quad \lambda = -\frac{\beta}{2}, \quad \mu = -1 - \frac{5\beta}{2}, \quad k_1 = -1, \quad k_2 = -3/2, \quad k_3 = -3, \quad k_4 = 3/2, \\ &k_5 = -3, \quad k_6 = -5/2, \quad k_7 = 1, \quad k_8 = 1/2, \quad \hat{\rho} = -19/4, \quad \hat{\Gamma} = -3/4, \quad \hat{\Delta} = -9/2, \\ &\Psi = -(\beta^2 - \beta + 8)/4, \quad \hat{\Psi} = \Psi + \beta/2, \quad \hat{\Omega} = \Omega - 27/2 \end{aligned}$$

hold in (2.1), then iterative scheme (1.3) defines a biparametric family of sixteenth-order methods satisfying the error equation below: with $e_n = x_n - \alpha$ for $n = 0, 1, 2, \ldots$

$$e_{n+1} = -\frac{1}{16}c_2c_3\{20c_2^2c_3 - 3c_3^2 + 2c_2c_4 + c_2^4(\beta - 2)\}\phi e_n^{16} + O(e_n^{17}), \quad (2.2)$$

where $c_2c_3c_4 \neq 0$ *and* $\phi = 36c_3^4 + 152c_2^3c_3c_4 - 38c_2c_3^2c_4 + 4c_2^2(-69c_3^3 + 3c_4^2 + 2c_3c5) + 12c_2^5c_4(\beta - 2) + 3c_2^8(\beta - 2)^2 + 2c_2^6c_3[\beta(2\beta^2 + \beta + 46) - 124] + c_2^4c_3^2[16\Omega - \beta(2\beta + 7) + 710]$ *and* β, Ω *are two free real parameters.*

Proof. Taylor series expansion of $f(x_n)$ about α up to sixteenth-order terms yields with $f(\alpha) = 0$:

$$f(x_n) = f'(\alpha)\left\{e_n + \sum_{i=2}^{16} c_i e_n^i + O(e_n^{17})\right\}. \tag{2.3}$$

For ease of notation, e_n will be denoted by e for the time being. With the aid of symbolic computation of Mathematica, a lengthy algebraic computation induces relations (2.4)–(2.9) below:

$$f'(x_n) = f'(\alpha)\left\{1 + \sum_{i=2}^{16} ic_i e^{i-1} + O(e^{16})\right\}. \tag{2.4}$$

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= e - c_2e^2 + 2(c_2^2 - c_3)e^3 - (4c_2^3 - 7c_2c_3 + 3c_4)e^4 + (8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e^5 \\ &\quad + H_6e^6 + H_7e^7 + H_8e^8 + \sum_{i=9}^{16} H_ie^i + O(e^{17}),\end{aligned} \tag{2.5}$$

where $H_i = H_i(c_2, c_3, c_4, c_5, H_6, H_7, \ldots, H_i)$ with three explicitly written coefficients $H_6 = -16c_2^5 + 52c_2^3c_3 - 33c_2c_3^2 - 28c_2^2c_4 + 17c_3c_4 + 13c_2c_5 - 5c_6$, $H_7 = 2(16c_2^6 - 64c_2^4c_3 - 9c_3^3 + 36c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2 - 2c_5) + 11c_3c_5 + c_2(-46c_3c_4 + 8c_6) - 3c_7)$, $H_8 = -64c_2^7 + 304c_2^5c_3 - 176c_2^4c_4 - 75c_3^2c_4 + 31c_4c_5 + c_2^3(-408c_3^2 + 92c_5) + 4c_2^2(87c_3c_4 - 11c_6) + 27c_3c_6 + c_2(135c_3^3 - 64c_4^2 - 118c_3c_5 + 19c_7) - 7c_8$.

$$\begin{aligned} y_n &= x_n - f(x_n)/f'(x_n) = \alpha + c_2e^2 - 2(c_2^2 - c_3)e^3 - (4c_2^3 - 7c_2c_3 + 3c_4)e^4 \\ &\quad - (8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e^5 - \sum_{i=6}^{16} H_ie^i + O(e^{17}). \end{aligned} \tag{2.6}$$

$$\begin{aligned} f(y_n) &= f'(\alpha)\left\{c_2e^2 + (2c_3 - 2c_2^2)e^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e^4 \right.\\ &\quad \left. - 2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e^5 + \sum_{i=6}^{16} \theta_ie^i + O(e^{17})\right\}, \end{aligned} \tag{2.7}$$

where $\theta_i = \theta_i(c_2, c_3, \ldots, c_i)$ with three explicitly written coefficients $\theta_6 = 12c_2^5 - 21c_2^3c_3 + 4c_2c_3^2 + 6c_2^2c_4 - H_6$, $\theta_7 = -32c_2^6 + 78c_2^4c_3 - 34c_2^2c_3^2 - 32c_2^3c_4 + 12c_2c_3c_4 + 8c_2^2c_5 - H_7$ and $\theta_8 = 48c_2^7 - 144c_2^5c_3 + 65c_2^4c_4 + 4c_2^3(27c_3^2 - 4c_5) + c_2(-12c_3^3 + 9c_4^2 + 16c_3c_5) - c_2^2(73c_3c_4 + 2H_6) - H_8$. Using K_f in (2.1), we obtain:

$$z_n = y_n - K_f(x_n)\frac{f(y_n)}{f'(x_n)} = \alpha + \sum_{i=4}^{16} L_ie^i + O(e^{17}), \tag{2.8}$$

where coefficients $L_i = L_i(c_2, c_3, \ldots, c_6, \lambda, \beta, \mu, H_6, H_7, \ldots, H_{16})$; for instance, $L_4 = -c_2c_3 + c_2^3(1 + 2\beta - \lambda + \mu)$ and $L_5 = -2c_3^2 - 2c_2c_4 + 2c_2^2c_3(4 + 6\beta - 3\lambda + 3\mu) - c_2^4(4 + 2\beta^2 - 8\lambda + 6\mu + \beta(12 - \lambda + \mu))$.

$$f(z_n) = f'(\alpha)\left\{L_4e^4 + L_5e^5 + L_6e^6 + L_7e^7 + (c_2L_4^2 + L_8)e^8 + \sum_{i=9}^{16} Z_ie^i + O(e^{17})\right\}, \tag{2.9}$$

where $Z_i = Z_i(L_4, L_5, \ldots, L_{16})$. Using relations (2.3)–(2.9), we can further express u_n, v_n and w_n in terms of $\beta, \lambda, \mu, a, b, c, d, \gamma, \sigma$ and $c_j (j = 2, 3, \ldots, 16)$ to compute s_n in (1.3) with H_f in (2.1):

$$s_n = \alpha + S_4e^4 + S_5e^5 + S_6e^6 + S_7e^7 + S_8e^8 + \sum_{i=9}^{16} S_ie^i + O(e^{17}), \tag{2.10}$$

where $S_i (i = 4, 5, \ldots, 16)$ are multivariate polynomials in $H_k (6 \le k \le 16)$, $L_\nu (4 \le \nu \le 16)$, $\beta, \lambda, \mu, a, b, c, d, \gamma, \sigma$ and $c_j (j = 2, 3, 4, 5)$ or $\lambda, \beta, \mu, a, b, c, d, \gamma, \sigma$ and $c_j (j = 2, 3, \ldots, 16)$; for instance, $S_4 = -c_2c_3 - L_4 + c_2^3(1 + 2\beta - \lambda + \mu) = 0$ is satisfied with $L_4 = -c_2c_3 + c_2^3(1 + 2\beta - \lambda + \mu)$ in (2.8) and

$$S_5 = (2 - a + c)c_2^2\left\{-c_3 + c_2^2(1 + 2\beta - \lambda + \mu)\right\}. \tag{2.11}$$

We require $S_5 = S_6 = \cdots = 0$ to achieve maximal order of convergence. Simplifying S_6, S_7, we obtain:

$$\begin{aligned} &c_2\{-c_3 + c_2^2(1+2\beta-\lambda+\mu)\}\{c_3(-1+b-d) + c_2^2[2+2a+d+2d\beta-d\lambda+d\mu-b(1+2\beta-\lambda+\mu)]\}, \\ &-c_2^2\{2(1+a)c_2^2-c_3\}\{c_2^2[4b-3\beta-2\gamma-2\mu+2\sigma+2a(1+2b+\beta-\gamma+\sigma)]+c_3(-a-2b+\gamma-\sigma)\}. \end{aligned} \tag{2.12}$$

This yields the relations independently of c_j's as follows:

$$c = a-2, \qquad d = b-1, \qquad \lambda = 2\beta+\mu-2a-1, \qquad \gamma = a+2b+\sigma, \qquad \mu = -a^2+(a-3/2)\beta. \tag{2.13}$$

Coefficient S_8 with (2.13) becomes:

$$S_8 = \frac{1}{2}c_2\{2(1+a)c_2^2-c_3\}\{2c_2c_4+2bc_3^2-4c_2^2c_3[4b+a(2+b)+\sigma]+c_2^4\kappa\}, \tag{2.14}$$

with $\kappa = -4+14a^2+24b-3\beta+8\sigma+4a(3+6b-\beta+2\sigma)$. Observe that S_8 can no longer be set to zero independently of c_j's, due to the terms c_j's with their factors free from control parameters.

Using relations (2.3)–(2.14), we can further express u_n, v_n, w_n and t_n in terms of a, b, β, σ and $c_j (j = 2, 3, \ldots, 16)$ by symbolic computation of Mathematica to compute x_{n+1} in (1.3) with W_f in (2.1) as follows:

$$x_{n+1} = \alpha + h_9e^9 + h_{10}e^{10} + h_{11}e^{11} + h_{12}e^{12} + h_{13}e^{13} + h_{14}e^{14} + h_{15}e^{15} + h_{16}e^{16} + O(e^{17}), \tag{2.15}$$

where $h_i (i = 9, 10, \ldots, 16)$ are multivariate polynomials in $\lambda, \beta, \mu, a, b, \gamma, \sigma, k_j (j = 1, 2, \ldots, 8), \rho, \Gamma, \Psi, \Delta, \Omega, \hat{\rho}, \hat{\Gamma}, \hat{\Psi}, \hat{\Delta}, \hat{\Omega}, S_j (j = 8, 9, \ldots, 16)$ and $c_j (j = 2, 3, \ldots, 16)$; for instance,

$$h_9 = (2-k_1+k_5)c_2S_8. \tag{2.16}$$

We impose conditions $h_9 = h_{10} = \cdots = h_{14} = h_{15} = 0$ and $h_{16} \neq 0$ independently of c_j's so that (1.3) has sixteenth-order convergence. By further simplifying $h_{10} = \cdots = h_{14} = h_{15}$, we obtain the following:

$$\begin{aligned} h_{10} &= S_8\{2[1+k_1-(1+a)k_2+k_6+ak_6]c_2^2+(-1+k_2-k_6)c_3\}, \\ h_{11} &= c_2S_8\{2(1+a)c_2^2-c_3\}(a+2k_2-k_3+k_7), \\ h_{12} &= \frac{1}{2}\frac{c_2^2\{c_3-2(1+a)c_2^2\}^2}{4(1+a)c_2^3-2c_2c_3}\omega A \\ h_{13} &= \frac{1}{4}c_2^2\{2(1+a)c_2^2-c_3\}\omega B, \\ h_{14} &= \frac{1}{8}c_2c_3\{2bc_3^2+2c_2c_4+c_2^4(-2+\beta)-4c_2^2c_3(-2+3b+\sigma)\}\omega C, \\ h_{15} &= -\frac{1}{8}c_2c_3\{-3c_3^2+2c_2c_4+c_2^4(\beta-2)+2c_2^2c_3(13-2\sigma)\}D, \end{aligned} \tag{2.17}$$

where $\omega = 2bc_3^2+2c_2c_4-4c_2^2c_3\{4b+a(2+b)+\sigma\}+c_2^4\{-4+24b-3\beta+8\sigma+2a(6+7a+12b-2\beta+4\sigma)\}$ and $A = 2(1-k_4+k_8)c_2c_4+2c_3^2\{k_2+b(-k_4+k_8)\}+2c_2^2c_3\{-1-4k_2-2k_3-2a(1+k_2)+2(k_4-k_8)[(4b+a(2+b)+\sigma)]\}+c_2^4\{6a^2+8k_2+8k_3+4a[2(1+k_2+k_3)-3\beta]+(-k_4+k_8)[-4+24b-3\beta+8\sigma+2a(6+7a+12b-2\beta+4\sigma)]\}$, $B = 2(-2+a+2k_4)c_2c_4+c_3^2\{3+2b(-2+a+2k_4)-2\rho+2\hat{\rho}\}+c_2^2c_3\{-4[a^2(2+b)+8b(-1+k_4)+a(-2+4k_4+2b(1+k_4)-2\rho+\sigma)-2(-1+\rho+\sigma-k_4\sigma)]-8(1+a)\hat{\rho}\}+c_2^4\{14a^3+a[-20+24b(-1+2k_4)+5\beta-16\rho-8k_4(-3+\beta-2\sigma)-8\sigma]+4a^2(-3+6b+7k_4-\beta-2\rho+2\sigma)+2[6+24b(-1+k_4)+3\beta-4\rho-8\sigma+k_4(-4-3\beta+8\sigma)]+8(1+a)^2\hat{\rho}\}$, $C = 2(3+2b)c_2c_3c_4+2c_3^3(b+2b^2-2\Gamma+2\hat{\Gamma})+4c_2^3c_4(\Delta-\hat{\Delta}-3)+2c_2^6(-2+\beta)(\Delta-\hat{\Delta}-3)-4c_2^2c_3^2[2\rho+\sigma-4+b(3+b+\hat{\Delta}-\Delta+2\sigma)]+c_2^4c_3[\beta-54+2b(\beta-12\Delta+34)-8\Delta(\sigma-2)+24\sigma+8(b+\sigma-2)\hat{\Delta}+4(\hat{\Psi}-\Psi)]$, $D = 4c_2^3c_4(2\Delta+3)+2c_2^6(\beta-2)(2\Delta+3)+2c_2c_3c_4(3-2\sigma)+c_3^3(6\sigma-8\Gamma-3)-c_2^4c_3(-134+\beta(2\beta-5)-104\Delta+20\sigma+2(\beta+8\Delta)\sigma+8\Psi)+2c2^2c_3^2(4\sigma(\sigma-9)-6\Delta-2\Omega+2\hat{\Omega}+63)$.

Solving (2.16) and (2.17) independently of c_j's yields relations among the constant parameters:

$$\begin{aligned} &a = -1, \quad b = -\frac{3}{2}, \quad c = -3, \quad d = -\frac{5}{2}, \quad \gamma = -\frac{5}{2}, \quad \sigma = \frac{3}{2}, \quad \rho = -\frac{13}{4}, \quad \Gamma = \frac{3}{4}, \\ &\Delta = -\frac{3}{2}, \quad \lambda = -\frac{\beta}{2}, \quad \mu = -1-\frac{5\beta}{2}, \quad k_1 = -1, \quad k_2 = -3/2, \quad k_3 = -3, \quad k_4 = 3/2, \\ &k_5 = -3, \quad k_6 = -5/2, \quad k_7 = 1, \quad k_8 = 1/2, \quad \hat{\rho} = -19/4, \quad \hat{\Gamma} = -3/4, \quad \hat{\Delta} = -9/2, \\ &\Psi = -(\beta^2-\beta+8)/4, \quad \hat{\Psi} = \Psi+\beta/2, \quad \hat{\Omega} = \Omega-27/2 \end{aligned} \tag{2.18}$$

with two free constant control parameters β, Ω.

Table 1
Asymptotic error constants for typical choices of β, Ω.

Case	Method	(β, Ω)	η
1	**Q1**	$(0, -355/8)$	$(1/8)\lvert c_2c_3(2c_2^4-20c_2^2c_3+3c_3^2-2c_2c_4)(\phi_1-\tau)\rvert$
2	**Q2**	$(-7/2, -355/8)$	$(1/32)\lvert c_2c_3(-11c_2^4+40c_2^2c_3-6c_3^2+4c_2c_4)(\phi_2-2\tau)\rvert$
3	**Q3**	$(2, -43)$	$(1/8)\lvert c_2c_3(20c_2^2c_3-3c_3^2+2c_2c_4)(\phi_3+\tau)\rvert$
4	**Q4**	$(2, -355/8)$	$(1/8)\lvert c_2c_3(20c_2^2c_3-3c_3^2+2c_2c_4)(\phi_4+\tau)\rvert$
5	**Q5**	$(0, 0)$	$(1/8)\lvert c_2c_3(2c_2^4-20c_2^2c_3+3c_3^2-2c_2c_4)(\phi_5-\tau)\rvert$

where $\tau = 2c_2^2(69c_3^3-3c_4^2-2c_3c_5)$,
$\phi_1 = 6c_2^8-124c_2^6c_3+18c_3^4-12c_2^5c_4+76c_2^3c_3c_4-19c_2c_3^2c_4$,
$\phi_2 = 363c_2^8/4-717c_2^6c_3+36c_3^4-66c_2^5c_4+152c_2^3c_3c_4-38c_2c_3^2c_4$,
$\phi_3 = 12c_2^6c_3-18c_3^4-76c_2^3c_3c_4+19c_2c_3^2c_4$,
$\phi_4 = 12c_2^6c_3+11c_2^4c_3^2-18c_3^4-76c_2^3c_3c_4+19c_2c_3^2c_4$,
$\phi_5 = 6c_2^8-124c_2^6c_3+355c_2^4c_3^2+18c_3^4-12c_2^5c_4+76c_2^3c_3c_4-19c_2c_3^2c_4$.

Table 2
Convergence for $f(x) = 2x^2+2x^3\cos(x^2-x+1)+3-i\sqrt{3}$ with $\alpha = (1+i\sqrt{3})/2$ and **Q1**.

n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert = \lvert x_n-\alpha\rvert$	$\left\lvert \frac{e_n}{e_{n-1}{}^{16}} \right\rvert$	η
0	$0.45+0.8i$	0.342685	0.0828212		
1	$\begin{pmatrix}0.500000000000011\\0.866025403784438\end{pmatrix}^*$	2.49924×10^{-15}	5.73366×10^{-16}	116.9947222	184.8880726
2	$\begin{pmatrix}0.500000000000000\\0.866025403784439\end{pmatrix}$	1.09951×10^{-241}	2.52244×10^{-242}	184.8880726	
3	$\begin{pmatrix}-0.500000000000000\\0.866025403784439\end{pmatrix}$	0.0×10^{-999}	0.0×10^{-1000}		

* $\begin{pmatrix}0.500000000000011\\0.866025403784438\end{pmatrix} = 0.500000000000011+0.866025403784438i$, $i=\sqrt{-1}$.

Table 3
Convergence for $f(x) = x^3\sin(x-1)+e^{-x}-1$ with $\alpha \approx 1.32262066495127$ and **Q5**.

n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert = \lvert x_n-\alpha\rvert$	$\left\lvert \frac{e_n}{e_{n-1}{}^{16}} \right\rvert$	η
0	1.5	0.841191	0.177379		
1	1.32262066492738	8.58078×10^{-11}	2.38901×10^{-11}	24.87534272	1030.915477
2	1.32262066495127	4.16880×10^{-167}	1.16065×10^{-167}	1030.915477	
3	1.32262066495127	0.0×10^{-1000}	0.0×10^{-999}		

We restore notation e back to e_n in (2.15) and compute h_{16} using (2.18), $L_5, L_6, L_7, L_8, L_9, L_{10}$ in (2.8) and $S_8, S_9, S_{10}, S_{11}, S_{12}$ associated with (2.10) after simplification as follows:

$$h_{16} = -\frac{1}{16}c_2c_3\{20c_2^2c_3-3c_3^2+2c_2c_4+c_2^4(\beta-2)\}\phi, \tag{2.19}$$

where $\phi = 36c_3^4+152c_2^3c_3c_4-38c_2c_3^2c_4+4c_2^2(-69c_3^3+3c_4^2+2c_3c5)+12c_2^5c_4(\beta-2)+3c_2^8(\beta-2)^2+2c_2^6c_3[\beta(2\beta^2+\beta+46)-124]+c_2^4c_3^2[16\Omega-\beta(2\beta+7)+710]$. This completes the proof with two free parameters β, Ω. □

Table 1 lists some choices of parameters β, Ω and the corresponding asymptotic error constant $\eta = |h_{16}|$.

3. Numerical results and discussions

By Mathematica [12] programming based on scheme (1.3), numerical experiments have been performed with the minimum 1000 precision digits, being large enough to minimize round-off errors as well as to clearly observe the computed asymptotic error constants requiring small-number divisions. To obtain accurate asymptotic error constants, the zero α was found with 1050 significant digits; the error bound $\epsilon = 0.5\times10^{-250}$ was used. The values of x_0 were selected closely to α for convergence. The computed asymptotic error constant agrees up to 10 significant digits with the theoretical one. The computed zero is rounded to be accurate up to 250 significant digits, although being displayed up to 15 significant digits.

Test functions $f(x) = 2x^2+2x^3\cos(x^2-x+1)+3-i\sqrt{3}$ with **Q1** and $f(x) = x^3\sin(x-1)+e^{-x}-1$ with **Q5** demonstrated sixteenth-order convergence. Tables 2 and 3 list iteration indexes $n, x_n, |f(x_n)|, |e_n|$, computational asymptotic error constants $|\frac{e_n}{e_{n-1}{}^{16}}|$ and the theoretical asymptotic error constant η.

Convergence behavior was verified for additional test functions that are listed below:

$$f_1(x) = (2+x^3)\cos\left(\frac{\pi x}{2}\right)+\log(x^2+2x+2), \quad \alpha=-1,\ x_0=-0.93$$

Table 4
Comparison of CPU times for high-order methods.

f	x_0	CPU time(s)						
		N1	**N2**	**N3**	**Q1**	**Q2**	**Q4**	**Q5**
f_1	−0.93	267.984	369.859	268.328	**57.188**	91.328	78.578	61.156
f_2	1.40	499.328	693.469	498.859	102.796	163.141	133.031	**102.11**
f_3	−1.16	2735.38	3823.84	2582.25	568.093	885.032	686.765	**523.782**
f_4	0.07	164.203	226.328	169.344	31.61	50.531	39.984	**31.313**
f_5	−1.84	404.312	602.063	409.515	80.531	126.344	103.016	**79.813**
f_6	0.98−1.36i	375.531	638.094	419.609	67.672	106.672	86.844	**67.125**
f_7	1.6	236.765	320.531	242.000	46.093	71.141	58.344	**45.578**

Table 5
Comparison of $|x_n - \alpha|$ for high-order iterative methods.

$f(x)$	x_0	$\|x_n-\alpha\|$	**N1**	**N2**	**N3**	**Q1**	**Q2**	**Q4**	**Q5**
f_1	−0.93	$\|x_1-\alpha\|$	8.91e−11*	2.76e−10	4.83e−10	1.45e−10	3.55e−10	1.07e−10	2.22e−10
		$\|x_2-\alpha\|$	7.98e−151	3.92−142	7.56e−138	1.05e−148	8.18e−142	**7.69e−152**	1.42e−145
		$\|x_3-\alpha\|$	0.e−1000	0.e−1000	0.e−1000	0.e−1000	0.e−1000	0.e−1000	0.e−1000
f_2	1.40	$\|x_1-\alpha\|$	5.13e−10	9.03e−9	4.31e−8	1.02e−10	4.97e−11	4.62e−09	1.31e−11
		$\|x_2-\alpha\|$	1.18e−145	5.19e−125	9.46e−114	5.26e−157	1.31e−161	7.91e−131	**4.57e−171**
		$\|x_3-\alpha\|$	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999
f_3	−1.16	$\|x_1-\alpha\|$	8.19e−13	4.07e−12	8.35e−12	2.21e−13	1.12e−12	1.01e−13	3.11e−13
		$\|x_2-\alpha\|$	3.36e−189	4.19e−177	1.12e−171	2.04e−201	5.42e−189	**7.30e−209**	4.82e−199
		$\|x_3-\alpha\|$	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999
f_4	0.07	$\|x_1-\alpha\|$	4.80e−14	3.01e−13	6.62e−13	2.29e−15	2.22e−14	6.01e−16	2.73e−15
		$\|x_2-\alpha\|$	1.18e−207	7.12e−194	5.91e−188	3.81e−232	8.52e−215	**2.30e−241**	2.44e−231
		$\|x_3-\alpha\|$	7.27.e−1220	1.26e−1190	5.52e−1173	4.30e−1474	0.0e−1856	0.0e−1962	0.0e−1922
f_5	−1.84	$\|x_1-\alpha\|$	2.29e−12	9.68e−12	2.62e−11	5.44e−13	1.62e−12	1.99e−13	2.47e−12
		$\|x_2-\alpha\|$	6.84e−188	1.42e−177	2.06e−170	2.53e−197	1.13e−189	**2.48e−204**	1.32e−186
		$\|x_3-\alpha\|$	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999
f_6	0.98 −1.36i	$\|x_1-\alpha\|$	2.00e−13	9.22e−13	1.96e−12	3.56e−14	1.18e−13	2.24e−14	6.91e−14
		$\|x_2-\alpha\|$	8.30e−196	1.91e−184	8.87e−179	8.65e−209	8.11e−200	**7.13e−213**	7.34e−204
		$\|x_3-\alpha\|$	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999
f_7	1.6	$\|x_1-\alpha\|$	1.06e−12	6.02e−12	1.79e−11	9.15e−15	9.12e−14	3.59e−14	9.82e−14
		$\|x_2-\alpha\|$	3.72e−192	1.60e−179	1.48e−171	**1.50e−227**	2.89e−210	3.37e−217	1.39e−209
		$\|x_3-\alpha\|$	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999	0.e−999

* 8.91e−11 denotes 8.91×10^{-11}.

$$f_2(x) = x^5 + x^4 + 4x^2\cos(1+x+x^2) - 15, \quad \alpha \approx 1.52576722495234,\ x_0 = 1.40$$
$$f_3(x) = 1 + xe^{x^{-1}} + 3x\cos x - x^2\sin x, \quad \alpha \approx -1.05559464153326,\ x_0 = -1.16$$
$$f_4(x) = e^x \sin x + \log(1+x^2), \quad \alpha = 0,\ x_0 = 0.07$$
$$f_5(x) = \sqrt{x^4+8}\,\sin\left(\frac{\pi}{x^2+2}\right) + \frac{x^3}{x^4+1} - \sqrt{6} + \frac{8}{17}, \quad \alpha = -2,\ x_0 = -1.84$$
$$f_6(x) = 1 - \cos[(x-1)^2+2] + \frac{\log[(x-1)^2+3]}{x}, \quad \alpha = 1 - i\sqrt{2}, \quad x_0 = 0.98 - 1.36i,\ i = \sqrt{-1}$$
$$f_7(x) = x^3 + \cos\left(\frac{\pi}{x^2}\right)\log(x^2-2) - 3\sqrt{3}, \quad \alpha \approx 1.73205080756888,\ x_0 = 1.6,$$

with $\log z (z \in \mathbb{C})$ representing a principal analytic branch such that $-\pi \le \mathrm{Im}(\log z) < \pi$.

In Table 4, CPU times are displayed for the listed methods. Indeed, the CPU time of (1.1) is greatly increased by a factor of about 5, being compared with that of (1.3). Table 5 lists the values of $|x_n - \alpha|$ for **N1**, **N2**, **N3** and **Q1**, **Q2**, **Q4**, **Q5**, where **N1**, **N2**, **N3** correspond to the values of $A = 0, 1, 2$ in (1.1), respectively. During the experiments, **Q4** has shown best accuracy for f_1, f_3, f_4, f_5, f_6, while **Q5** for f_2, and **Q1** for f_7. The corresponding efficiency index is $16^{1/5} \approx 1.741101$ better than $8^{1/4} \approx 1.68179$ for eighth-order methods [2,3,7,10] and better than $\sqrt{2}$ for Newton's method.

Although limited to the particular set of chosen test functions, most of the proposed methods have shown better performance than **N1**, **N2** and **N3**. They can be extended for optimal convergence order of 32 in accordance with the conjecture of Kung–Traub.

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